# Riemannian Median and Its Applications for Orientation Distribution Function Computing

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## Introduction

The geometric median is a classic robust estimator of centrality for data in Euclidean spaces, and it has been generalized in analytical manifold in [1]. Recently, an intrinsic Riemannian framework for Orientation Distribution Function (ODF) was proposed for the calculation in ODF field [2]. In this work, we prove the unique existence of the Riemannian median in ODF space. Then we explore its two potential applications, median filtering and atlas estimation

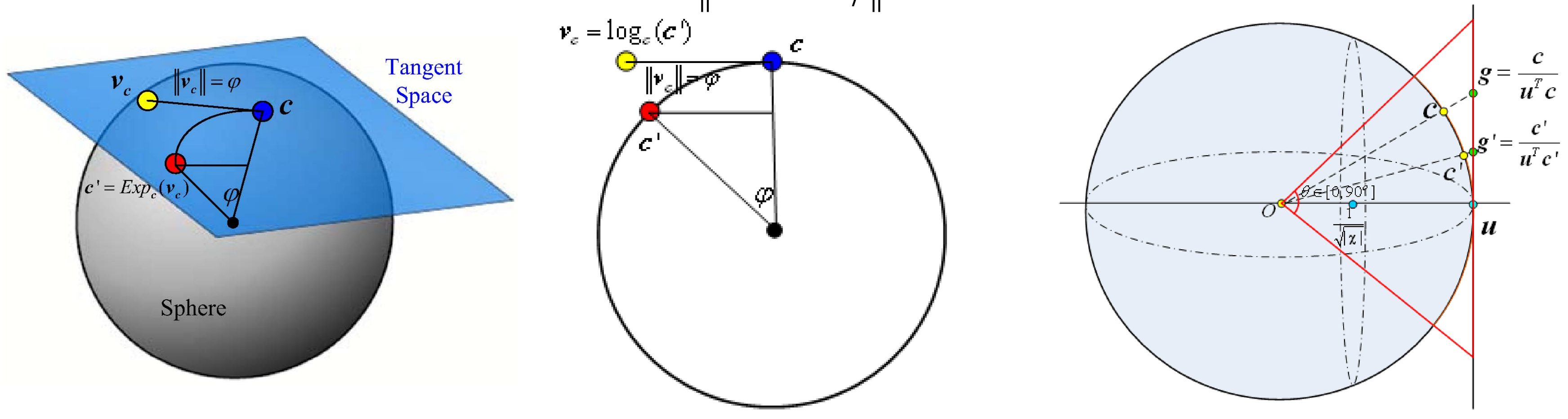
## Riemannian framework for ODFs

**I. PDF family via orthonormal basis representation [2]**

$$p(\mathbf{x}\,|\,\mathbf{c}) = \left(\sum_{i=1}^{K} c_i B_i(\mathbf{x})\right)^2 \qquad \sum_{i=1}^{K} c_i^2 = 1 \qquad \sum_{i=1}^{K} c_i B_i(\mathbf{x}) \ge 0$$

**II. The intrinsic Riemannian framework for ODF computing [2]**

- Fisher metric [6]: $g_{ij} = 4\int_{S^{K-1}} \partial_i \sqrt{p(\mathbf{x}\,|\,\mathbf{c})}\,\partial_j \sqrt{p(\mathbf{x}\,|\,\mathbf{c})}\,d\mathbf{x} = 4\delta_{ij}$
- Geodesic: $d_{g_{ij}}(p(\cdot\,|\,\mathbf{c}), p(\cdot\,|\,\mathbf{c}')) = d_{\delta_{ij}}(\mathbf{c},\mathbf{c}') = \arccos(\mathbf{c}^T\mathbf{c}')$
- Exponential map: $Exp_{\mathbf{c}}(\mathbf{v}_c) = \mathbf{c}' = \mathbf{c}\cos\varphi + \frac{\mathbf{v}_c}{\|\mathbf{v}_c\|}\sin\varphi$ , where $\varphi = \|\mathbf{v}_c\|$
- Logarithmic map: $Log_{\mathbf{c}}(\mathbf{c}') = \mathbf{v}_c = \frac{\mathbf{c}' - \mathbf{c}\cos\varphi}{\|\mathbf{c}' - \mathbf{c}\cos\varphi\|}\varphi$ , where $\varphi = \arccos(\mathbf{c}^T\mathbf{c}')$




**III. Properties of parameter space (PS) [2]**

- PS is a closed convex
- PS is contained in a convex cone with $90^0$
- The projection of any $\boldsymbol{u}$ on the uniform ODF $\boldsymbol{c}$ is more than $\frac{1}{\sqrt{4\pi}}$
- Weighted Riemannian mean uniquely exists $m_w = \arg\min_{f\in PS} \sum_{i=1}^{N} w_i d(f, f_i)^2$

## Weighted Riemannian Median

- **Definition**: $\mu_w = \arg\min_{f\in PS} \sum_{i=1}^{N} w_i d(f, f_i)$
- **Uniquely exists** if one of the two conditions is satisfied [1]
  - a) the sectional curvatures of M are non-positive
  - b) the sectional curvatures of M are bounded by $\Delta > 0$ and $diam(U) < \pi/(2\sqrt{\Delta})$ , where $U$ is the convex set which contains $\{x_i\}$ and $diam(U)$ is the diameter of $U$
- A Weighted Riemannian median **uniquely exists** in PS because of condition (b)
- No close form. Numerical algorithm [1]:

**Algorithm 1: Weighted Riemannian Median**

**Input**: $f_1, ..., f_N \in PS^K$, $\mathbf{w} = (w_1, ..., w_N)'$, $w_i \ge 0, i = 1, 2, ..., N$, $\sum_{i=1}^{N} w_i = 1$.

**Output**: $\mu_{\mathbf{w}}$, the *Weighted Frechet Mean*.

Initialization: $\mu_{\mathbf{w}}^{(0)} = \frac{\sum_{i=1}^{N} w_i f_i}{\|\sum_{i=1}^{N} w_i f_i\|}$, $k = 0$

Do

$q_i = w_i d(\mu_{\mathbf{w}}^{(k)}, x_i) / \sum_{i=1}^{n} w_i d(\mu_{\mathbf{w}}^{(k)}, x_i)$, $\quad v_{\mu_{\mathbf{w}}^{(k)}} = \sum_{i=1}^{N} q_i Log_{\mu_{\mathbf{w}}^{(k)}}(f_i)$

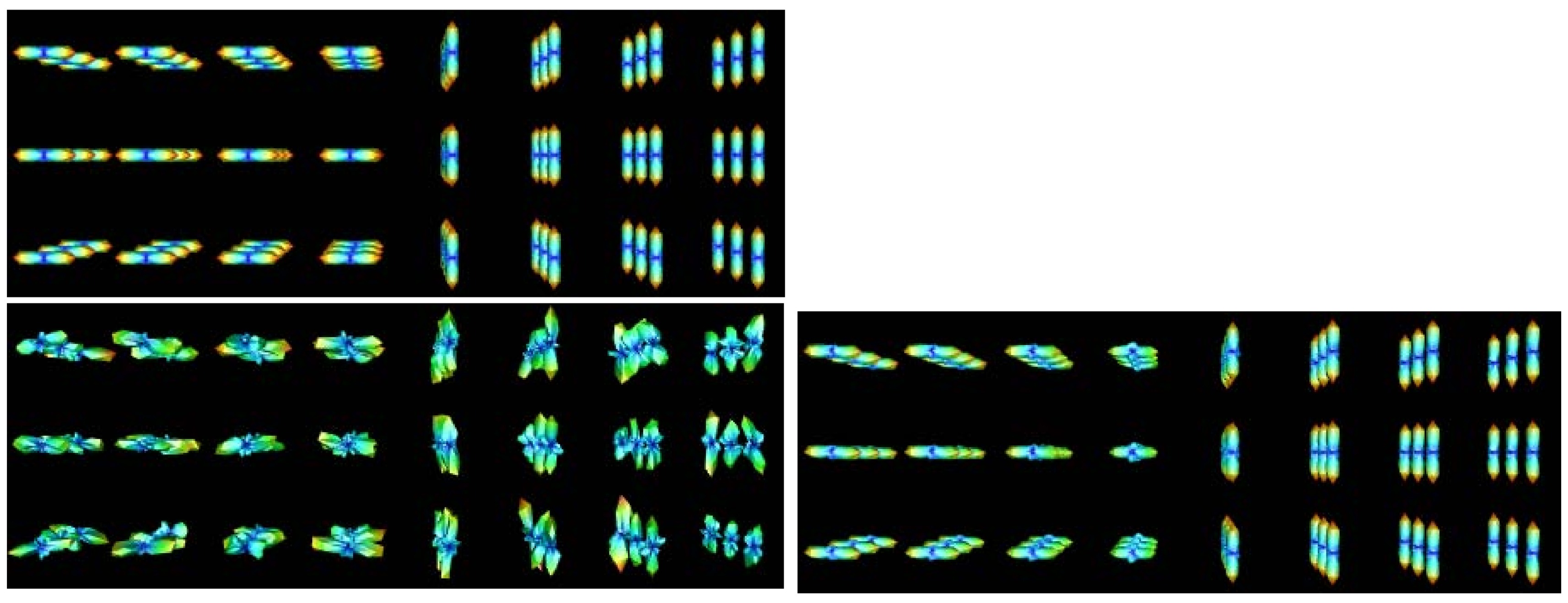
$\mu_{\mathbf{w}}^{(k+1)} = Exp_{\mu_{\mathbf{w}}^{(k)}}(v_{\mu_{\mathbf{w}}^{(k)}})$

$k = k + 1$

while $\|v_{\mu_{\mathbf{w}}^{(k)}}\| > \varepsilon$

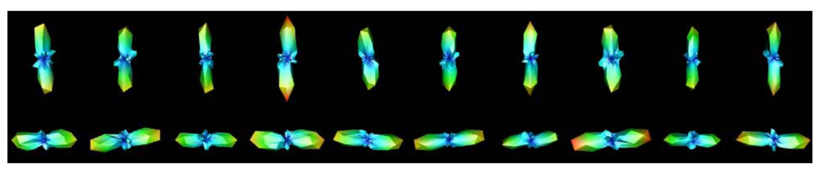
## Median Smoothing

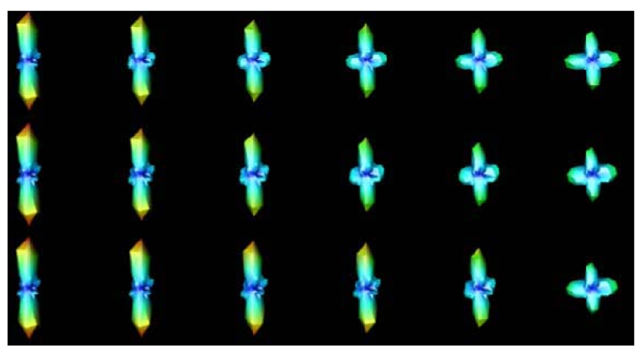
- Edge-preserving smoothing
- Experiment on synthetic data (Gaussian noise on tangent space)



## Atlas Estimation

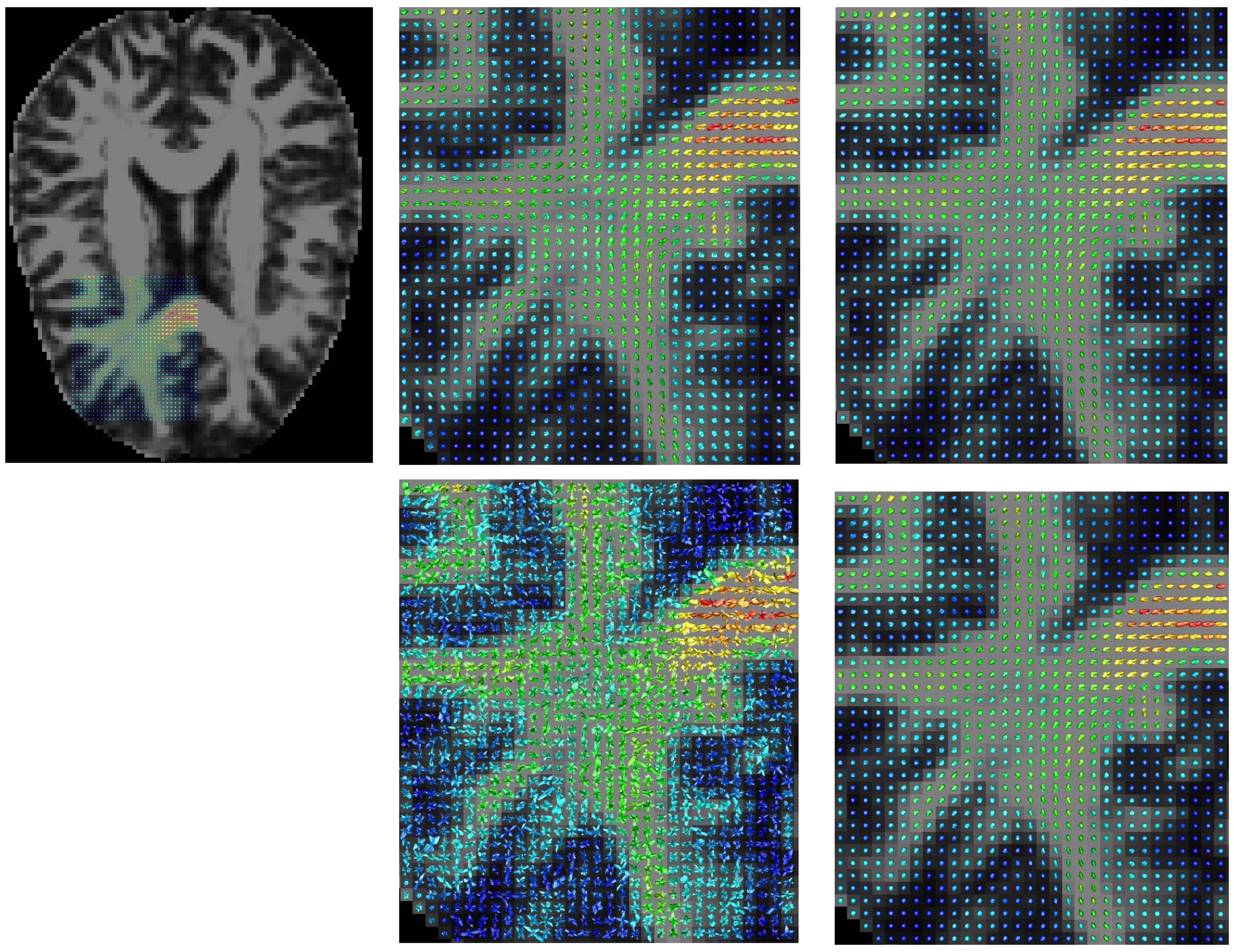
- Robust Statistics, breakdown point
- Experiment on outlier effect





Left: regular ODFs, outliers; Right: Euclidean mean, Riemannian mean, Riemannian median

- **Atlas Estimation on five subjects**
  - **Affine registration on all DWIs**
  - **Affine matrix from the registration on the image without diffusion via FSL**
  - **re-orientate the gradient directions via the finite strain (FS) [4]**



From left to right, first row: a slice of estimated atlas; ODF field in ROI for one subject; ODF field of the estimated atlas, second row: ODF field in ROI with Gaussian noise in Riemannian space (std=0.1) for one subject; ODF field of estimated atlas from the data with noise.

## References

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